Int. J. Heat Mass Transfer. Vol. 17, pp. 885–897. Pergamon Press 1974. Printed in Great Britain

GAS–LIQUID LAMINAR BOUNDARY-LAYER FLOWS WITH A WAVY PHASE-CHANGING INTERFACE

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(*Received* 15 *December* 1973)

Abstract—Laminar boundary-layer flows of gas and liquid having a wavy phase-changing interface are analytically investigated to predict the features of their disturbance field. The basic field is approximated to be of linear profiles and perturbed with small wavy disturbances. The rate of phase-change at the interface is disturbed proportionally to the 1/3rd power of wave number with the phase lag of 150° relative to the interface. The wavy disturbances have influence on the heat transfer at the interface by one order of magnitude less than on the skin-friction. Both coefficients are proportional to the 1/3rd power of wave number. The phase relation of shearing and normal stresses at the interface implies the same possibility of "water wave" instability of sheltering effect as that for isothermal cases, although the phase-change at the interface acts to alleviate such an effect both in amplitude and in phase relation.

NOMENCLATURE

c, wave velocity (U_∞);
c_p, specific heat;
f, amplitude of disturbance stream function;
g, amplitude of disturbance temperature;
$\boldsymbol{g}$, gravitational acceleration (U_∞^2/l_r);
h, amplitude of disturbance concentration;
k, amplitude of disturbance pressure;
L, latent heat of vaporization $[c_{p1}(T_{2\infty}-T_{1\infty})]$;
l_r, characteristic length $[=\sqrt{(\nu x/U_\infty)}]$;
p, pressure $(\rho_1 U_\infty^2)$;
T, temperature;
U, V, velocity components of base flow (U_∞);
u, v, disturbance velocity components (U_∞);
U^*, $=U-c$;
W, w, vapor concentration and its disturbance;
x, y, co-ordinates (l_r).

Greek symbols

α, wave number (l_r^{-1});
β_n, $=(-1)^n i\alpha_n U'_{n0}/\nu_n$, $\beta_{\kappa n}=(-1)^n i\alpha_n U'_{n0}/\kappa_n$, $\beta_\varepsilon=-i\alpha_1 U'_{10}/\varepsilon$;
δ, disturbance of interface elevation (l_r);
ε, diffusion coefficient of vapor $(U_\infty l_{r1})$;
Θ, θ, dimensionless temperature and its disturbance;
κ, thermal diffusivity $(U_\infty l_r)$;
λ, heat conductivity $(\rho_1 c_{p1} U_\infty l_r)$;
ν, kinematic viscosity $(U_\infty l_r)$;
ρ, density.

Subscripts

0, interface of gas and liquid;
1, gas side;
2, liquid side;
∞, infinity.

Values in parentheses or brackets indicate the reference unit.

INTRODUCTION

SIMULTANEOUS heat and mass transfer of gas and liquid flows having an interface at their boundary, where the phase state of the fluid is subjected to change, is basical and essential in engineering problems such as two-phase flow, drying process, etc. In a previous paper [1], an account was given of the solution of the boundary-layer equations for steady laminar flows with a plane phase-changing interface through which mass, momentum and energy were transferred continuously. It was shown that the phase-change at the interface takes considerable effect on the flow and thermal fields of gas and liquid.

Such an interface may be often found to be wavy. Concerning the mechanism by which kinematic energy may be transferred from a gas flow to a wavy liquid surface without mass and heat transfer, a number of works have been published. By Benjamin [2], Landahl [3], Lock [4] and Gupta *et al.* [5] for air flows over a flexible wall or a water surface, there are three essentially different types of instabilities possible, which can be identified with Tollmien–Schlichting waves, free surface waves and Kelvin–Helmholtz waves, respectively. An approach to such a problem is to solve directly and numerically a complete set of linearized equations of the whole system, as treated by Lock [4] and Feldman [6]. In cases that both gas and liquid motions can be separately unstable, however, stability analyses of this sort are extremely complicated so as to lose sight of physical reasoning. An alternative is to examine the features of stresses at the interface corresponding to the stationary wavy disturbances superposed on the mean field, as used by Benjamin [2] and Miles [7].

To examine the effect of wavy disturbances upon the laminar boundary-layer flows of gas and liquid with a phase-changing interface is thus prompted by two needs (i) to evaluate the friction and heat-transfer coefficients of flows having a wavy phase-changing surface and (ii) to give some informations of the instability problem of the phase-changing interface. In the present study, by taking the disturbance to be stationary relative to the wave on the surface and considering the perturbed system analytically in an approximate method, the aspects of the disturbance field are investigated.

LAMINAR BOUNDARY-LAYER FLOWS OF GAS AND LIQUID HAVING A WAVY PHASE-CHANGING INTERFACE

We consider, as in the previous study [1], the motion of a fully developed horizontal stream of incompressible fluid of gas over a liquid stream. At the boundary between the gas and liquid fluids, the fluid is submitted to change in the phase of state, from liquid to vapor (evaporation) or from vapor to liquid (condensation). At the point far from the gas–liquid interface, the gaseous fluid has the velocity U_∞ relative to the liquid stream and the temperature $T_{1\infty}$, and contains the vapor of the liquid by the mass fraction W_∞. The temperature of the liquid is $T_{2\infty}$. Properties of gas and liquid are denoted by subscripts 1 and 2, respectively. The flows are steady two-dimensional and fully developed to form laminar boundary layers with the common interface of a simple-harmonic wavy surface (Fig. 1).

In the absence of the waviness of interface, the fields of velocity, temperature and concentration are solved in the previous paper [1]. Denote them as (U, V), Θ and W, respectively. Here, the velocity components (U, V) are non-dimensionalized by U_∞ and the temperature by $(T_{2\infty} - T_{1\infty})$ as

$$\Theta_1 = \frac{T_1 - T_{1\infty}}{T_{2\infty} - T_{1\infty}}, \qquad \Theta_2 = \frac{T_2 - T_{2\infty}}{T_{1\infty} - T_{2\infty}}.$$

Ignoring the possible instability of the flow system with respect to time and accordingly taking the disturbance to be stationary relative to the wave on the

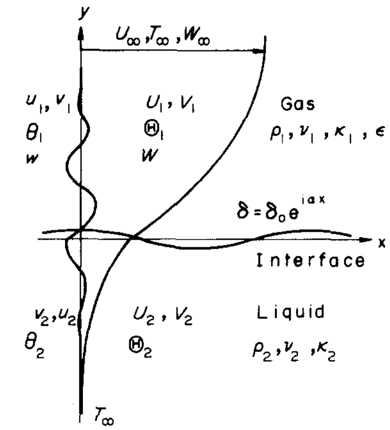


FIG. 1. Laminar boundary-layer flows of gas and liquid having a wavy phase-changing interface.

interface, we can study explicitly the interesting effect due to the presence of a wavy phase-changing interface. It is thus convenient to use a reference frame in which the wave on the interface is stationary, moving at speed c with the wave, so that the velocity parallel to the interface is $(U - c)$, which is henceforth denoted as

$$U^* = U - c.$$

In the case of the liquid stream dragged by an air flow, the wave speed of the interface would be of the order of the mean velocity of the liquid, so that we can approximate $U_0^* = O(U_0)$ which would be a very small fraction of the velocity of the air flow at infinity, since $U_0 \approx \{\rho_1^2 \nu_1 / (\rho_2^2 \nu_2)\}^{1/3}$ [1].

The elevation of the wavy interface is taken to be of the form

$$\delta = \delta_0 e^{i\alpha x}, \tag{1}$$

where the amplitude δ_0 is assumed to be small compared with the wave length $2\pi/\alpha$ so that $(\alpha\delta_0)^2$ is to be neglected. Let (u, v), θ and w be the associated disturbances of velocity, temperature and vapor-concentration, respectively. The equations of continuity, momentum, energy and concentration for the disturbance fields can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2}$$

$$\left(U^* \frac{\partial}{\partial x} + V \frac{\partial}{\partial y}\right)\begin{pmatrix}\omega \\ \theta \\ w\end{pmatrix} + \left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}\right)\begin{pmatrix}\Omega \\ \Theta \\ W\end{pmatrix} = (\nu, \kappa, \varepsilon)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\begin{pmatrix}\omega \\ \theta \\ w\end{pmatrix}, \tag{3}$$

where ν is the kinematic viscosity, κ the thermal diffusivity, ε the concentration diffusivity of the vapor, and

$$\Omega = \frac{\partial U}{\partial y} - \frac{\partial V}{\partial x}, \qquad \omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}.$$

Here, all variables representing a length are rendered non-dimensional by using, in the gas and liquid fluids respectively, the characteristic lengths l_{r1} and l_{r2};

$$l_{r1} = \sqrt{\left(\frac{\nu_1 x}{U_\infty}\right)}, \qquad l_{r2} = \sqrt{\left(\frac{\nu_2 x}{U_\infty}\right)}.$$

The introduction of these characteristic lengths allows similar solutions for the base fields of velocity, temperature and concentration with a plane phase-changing interface and thus for their perturbation (disturbance) fields with a wavy interface of small amplitude. Diffusivities ν, κ and ε are made dimensionless on the understanding that $l_r U_\infty$ is the unit.

Taking into account the equation of continuity, we put the disturbance velocity components as

$$u = if'(y)\,e^{i\alpha x}, \qquad v = \alpha f(y)\,e^{i\alpha x}. \tag{4}$$

With the boundary-layer approximation, equation (3) for ω is then reduced to

$$\alpha(iU^*F - iU''f) = \nu(F'' - \alpha^2 F), \quad F \equiv f'' - \alpha^2 f. \tag{5}$$

Writing the disturbance temperature and concentration as

$$\theta = g(y)\,e^{i\alpha x}, \qquad w = h(y)\,e^{i\alpha x}, \tag{6}$$

we have from equations (3)

$$\left.\begin{aligned} \alpha(iU^*g + \Theta' f) &= \kappa(g'' - \alpha^2 g), \\ \alpha(iU^*h + W'f) &= \varepsilon(h'' - \alpha^2 h). \end{aligned}\right\} \tag{7}$$

Consider now the boundary conditions of the disturbed field. At large distances from the interface, the disturbances must vanish so that we have

$$f_{\pm\infty} = f'_{\pm\infty} = 0, \qquad g_{\pm\infty} = 0, \qquad h_{\pm\infty} = 0. \tag{8}$$

Since the boundary conditions at the interface must be satisfied just at the wavy interface $y = \delta$, it would seem that the boundary conditions at the mean position of the interface $y = 0$ might require a further restriction on the wave amplitude so that, within the range of the distance normal to the interface less than the amplitude, the variations of U, Θ and W are negligibly small. This severe restriction can be avoided by the linearization of boundary conditions at the interface $y = \delta$ as pointed out by Benjamin [2] and Landahl [3]. A quantity $\Phi + \phi\,e^{i\alpha x}$ is linearized at $y = \delta_0\,e^{i\alpha x}$ in δ_0 as

$$\Phi_0 + \left\{\left(\frac{\partial \Phi_0}{\partial y}\right)_0 \delta_0 + \phi_0\right\} e^{i\alpha x},$$

where the subscript 0 refers to values at the mean position of the interface $y = 0$. This means that the corresponding disturbance amplitude of the quantity at the interface $\tilde{\phi}_0\,e^{i\alpha x}$ can be expressed as

$$\tilde{\phi}_0 = \phi_0 + \left(\frac{\partial \Phi}{\partial y}\right)_0 \delta.$$

The x-component of the disturbance velocity and the disturbance temperature must be continuous at the interface;

$$\tilde{u}_{10} = \tilde{u}_{20}, \tag{9$_u$}$$

$$\tilde{\theta}_{10} + \tilde{\theta}_{20} = 0, \tag{9$_t$}$$

where subscripts 10 and 20 denote the values at the interface of the upper and lower fluids respectively. The continuity relationships of the total mass of fluid and the mass of vapor at the interface give

$$\rho_1\left(\tilde{v}_1 - U^*\frac{\partial \delta_1}{\partial x_1}\right)_0 = \rho_2\left(\tilde{v}_2 - U^*\frac{\partial \delta_2}{\partial x_2}\right)_0, \tag{9$_c$}$$

$$\rho_1\left\{W\left(\tilde{v}_1 - U^*\frac{\partial \delta_1}{\partial x_1}\right) + \tilde{w}V_1 - \varepsilon\frac{\partial \tilde{w}}{\partial y_1}\right\}_0 = \rho_2\left(\tilde{v}_2 - U^*\frac{\partial \delta_2}{\partial x_2}\right)_0, \tag{9$_w$}$$

respectively, where ρ is the density, being assumed to be constant. The additional tangential stress due to the disturbance and the energy flux must also be continuous at the interface;

$$\rho_1\nu_1\left(\frac{\partial \tilde{u}_1}{\partial y_1} + \frac{\partial \tilde{v}_1}{\partial x_1}\right)_0 = \rho_2\nu_2\left(\frac{\partial \tilde{u}_2}{\partial y_2} + \frac{\partial \tilde{v}_2}{\partial x_2}\right)_0, \tag{9$_s$}$$

$$L\left(\tilde{v}_1 - U^*\frac{\partial \delta_1}{\partial x_1}\right)_0 = \lambda_1\left(\frac{\partial \tilde{\theta}_1}{\partial y_1}\right)_0 + \lambda_2\left(\frac{\partial \tilde{\theta}_2}{\partial y_2}\right)_0, \tag{9$_q$}$$

where L is the latent heat of vaporization and λ the heat conductivity, being non-dimensionalized by the quantities $c_{p1}(T_{2\infty} - T_{1\infty})$ and $\rho_1 c_{p1} U_\infty l_{r1,2}$ (c_p: specific heat), respectively.

The vapor at the interface can be assumed to have the state of saturation which holds Clausius–Clapeyron's relation

$$\frac{\mathrm{d}p_s}{\mathrm{d}T_0} = \frac{L}{R}\frac{T_{2\infty} - T_{1\infty}}{T_0}\frac{p_s}{T_0},$$

where p is the pressure non-dimensionalized by $\rho_1 U_\infty^2$ and R the gas constant divided by c_{p1}. This relation gives the disturbance concentration (or partial pressure) of the vapor at the interface

$$\tilde{w}_0 = W_0\left(\frac{\tilde{p}_s}{P_s} - \frac{\tilde{p}_{10}}{P_{10}}\right) = H_t\tilde{\theta}_{10} - H_p(\tilde{p}_{10} - g_1\delta_1), \tag{9$_e$}$$

where $\boldsymbol{g}_1$ is the gravitational acceleration non-dimensionalized by U_∞^2/l_{r1} and

$$H_t = W_0 \frac{L}{R}\left(\frac{T_{2\infty}-T_{1\infty}}{T_0}\right)^2, \qquad H_p = \frac{W_0}{P_{10}}.$$

By equations (4) and (6), these boundary conditions at the interface are rewritten in (f, g, h) expression as follows;

$$(f_1' - iU_1'\delta_1)_0 = (f_2' - iU_2'\delta_2)_0, \qquad (10)_u$$

$$(g_1 + \Theta_1'\delta_1)_0 + (g_2 + \Theta_2'\delta_2)_0 = 0, \qquad (10)_t$$

$$\rho_1\alpha_1(f_1 - iU^*\delta_1)_0 = \rho_2\alpha_2(f_2 - iU^*\delta_2)_0, \qquad (10)_c$$

$$\alpha_1(W_0-1)(f_1 - iU^*\delta_1)_0 + V_{10}(h + W'\delta_1)_0 - \varepsilon(h' + W''\delta_1)_0 = 0, \qquad (10)_w$$

$$\rho_1\nu_1(f_1'' - iU_1''\delta_1 + \alpha_1^2 f_1)_0 = \rho_2\nu_2(f_2'' - iU_2''\delta_2 + \alpha_2^2 f_2)_0, \qquad (10)_s$$

$$L\alpha_1(f_1 - iU^*\delta_1)_0 = \lambda_1(g_1' + \Theta_1''\delta_1)_0 + \lambda_2(g_2' + \Theta_2''\delta_2)_0, \qquad (10)_q$$

$$(h + W'\delta_1)_0 = H_t(g_1 + \Theta_1'\delta_1)_0 - H_p(k_1 - \boldsymbol{g}_1\delta_1)_0 \qquad (10)_e$$

where the primes indicate derivatives with respect to y, being non-dimensionalized by l_{r1} or l_{r2}. $\tilde{v}_0$ and $\tilde{p}_0$ are approximated by v_0 and p_0, respectively. The wave number α and the amplitude of the wavy interface δ_0 have the relation that

$$\alpha_1\delta_{10} = \alpha_2\delta_{20}, \qquad \alpha_1 x_1 = \alpha_2 x_2 = \alpha x. \qquad (11)$$

k is the amplitude of the disturbance pressure defined by

$$p = k\,e^{i\alpha x}, \qquad (12)$$

which is obtainable from the equation of motion in the x-direction;

$$\alpha k = \nu\{f''' - iU'''\delta - \alpha^2(f' - iU'\delta)\} - i\alpha U^*(f' - iU'\delta) - V(f'' - iU''\delta) + i\alpha U'f. \qquad (13)$$

The balance of normal stresses at the interface may give an additional relation of the pressure (including the effect of gravity and surface tension). We consider only the disturbances stationary relative to the wave on the interface so that there exists reaction for the disturbance flow to maintain the specified wavy disturbance. Without the external forces corresponding to such reaction, the wave number should be determined by an eigen-value problem of the system. If we take the normal stress condition as the external necessity of maintaining the disturbance of wave number α, the disturbance pressure is then determined as a subsequent quantity of the resulted disturbance field.

We have thus two fourth-order differential equations for f and two second-order for g and one second-order for h with seven boundary conditions at infinity and seven at the interface. In view of the boundary conditions at infinity, f, g and h can be written with constants A_n, B_n, C_n and D as

$$\left.\begin{aligned} f_n &= A_n f_{an} + B_n f_{bn}, \\ g_n &= A_n g_{an} + B_n g_{bn} + C_n g_{cn}, \quad (n = 1, 2) \\ h &= A_1 h_a + B_1 h_b + D h_d, \end{aligned}\right\} \qquad (14)$$

where f_a and f_b are the solutions of equations (5), g_a, g_b, h_a and h_b the solutions of equations (7) corresponding to f_a and f_b, respectively, and g_c and h_d those of equation (7) with $f \equiv 0$. The boundary conditions at the interface can be expressed in the form

$$a_{m1}A_1 + b_{m1}B_1 + a_{m2}A_2 + b_{m2}B_2 + c_{m1}C_1 + c_{m2}C_2 + d_m D + e_m\delta_0 = 0, \qquad (m = u, s, c, t, q, w, e) \quad (15)$$

where a_{mn}, b_{mn}, c_{mn}, d_m and e_m are functions of the values of f, g and h at the interface (see Appendix). The solution of equation (15) gives A_n, B_n, C_n and D corresponding to δ_0. Since equation (15) is linear, we can set $\delta_0 = 1$ for simplicity; thus, values of A_n, B_n, C_n and D are to be interpreted finally as a multiple of δ_0.

In order to examine quantitatively the effect of the wavy disturbance, we proceed to an analytical treatment of the problem with an approximation of linear profiles for the mean field. Since the disturbances diminish very rapidly with increasing the distance from the interface, the region where the magnitude of disturbances is significant in comparison with their values at the interface can be assumed to be largely covered with the linear profile region of the mean field, over which we can assume approximately

$$U = U_0 + U_0'y, \quad \Theta = \Theta_0 + \Theta_0'y, \quad W = W_0 + W_0'y. \qquad (16)$$

With this linear distribution of the mean velocity, equation (5) can be reduced to

$$F'' + \frac{\alpha}{i\nu}U_0'\left(y + \frac{U_0^* - i\alpha\nu}{U_0'}\right)F = 0. \qquad (17)$$

Defining a co-ordinate z as

$$z = y + z_0, \quad z_0 \equiv (U_0^* - i\alpha\nu)/U_0', \qquad (18)$$

we can rewrite the above equation for the upper fluid,

$$F_1'' + \beta_1 z_1 F_1 = 0, \quad \beta_1 \equiv -i\alpha_1 U_{10}'/\nu_1, \qquad (17)'$$

of which the solution is a linear combination of the functions $F^{(1)}(z)$ and $F^{(2)}(z)$;

$$F^{(1,2)}(z) = \sqrt{z}\,H_{1/3}^{(1,2)}(\tfrac{2}{3}\sqrt{\beta}z^{3/2}),$$

where $H_{1/3}^{(1)}$ and $H_{1/3}^{(2)}$ are the Hankel functions of order one-third of the first and second kind, respectively.

Accordingly, equation (5) for the upper fluid

$$f_1''-\alpha_1^2 f_1 = F_1 \tag{19}$$

subject to the boundary condition $f_1 = f_1' = 0$ at $y_1 = \infty$ has the solution of

$$f_1 = A_1\,\mathrm{e}^{-\alpha_1 y_1} + B_1\left[\mathrm{e}^{\alpha_1 y_1} - \frac{2\int_{z_{10}}^{z_1} F_1^{(2)}(\xi)\sinh\{\alpha_1(z_1-\xi)\}\,\mathrm{d}\xi}{\mathrm{e}^{\alpha_1 z_{10}}\int_{z_{10}}^{\infty} F_1^{(2)}(\xi)\,\mathrm{e}^{-\alpha_1\xi}\,\mathrm{d}\xi}\right]. \tag{20}$$

For the lower fluid ($y < 0$), denoting

$$\hat{z}_2 = \hat{y}_2+\hat{z}_{20},\quad \hat{y}_2 = -y_2,\quad \hat{z}_{20} = -z_{20},\quad \beta_2 \equiv i\alpha_2 U_{20}'/\nu_2, \tag{21}$$

we can take the same manipulation as the above to obtain

$$f_2 = A_2\,\mathrm{e}^{-\alpha_2\hat{y}_2} + B_2\left[\mathrm{e}^{\alpha_2\hat{y}_2} - \frac{2\int_{\hat{z}_{20}}^{\hat{z}_2} F_2^{(1)}(\xi)\sinh\{\alpha_2(\hat{z}_2-\xi)\}\,\mathrm{d}\xi}{\mathrm{e}^{\alpha_2\hat{z}_{20}}\int_{\hat{z}_{20}}^{\infty} F_2^{(1)}(\xi)\,\mathrm{e}^{-\alpha_2\xi}\,\mathrm{d}\xi}\right]. \tag{22}$$

Values of f, f' and f'' at the interface are thus given by

$$\left.\begin{aligned} f_{n0} &= A_n+B_n,\\ f_{n0}' &= (-1)^n\alpha_n(A_n-B_n), \qquad (n=1,2)\\ f_{n0}'' &= \alpha_n^2\{A_n+(1-\tau_n)B_n\},\end{aligned}\right\} \tag{23}$$

where

$$\tau_1 = \frac{2F_1^{(2)}(z_{10})}{\alpha_1\,\mathrm{e}^{\alpha_1 z_{10}}\int_{z_{10}}^{\infty} F_1^{(2)}(\xi)\,\mathrm{e}^{-\alpha_1\xi}\,\mathrm{d}\xi},$$

$$\tau_2 = \frac{2F_2^{(1)}(\hat{z}_{20})}{\alpha_2\,\mathrm{e}^{\alpha_2\hat{z}_{20}}\int_{\hat{z}_{20}}^{\infty} F_2^{(1)}(\xi)\,\mathrm{e}^{-\alpha_2\xi}\,\mathrm{d}\xi}. \tag{24}$$

Next, we consider the temperature field. Equation (7) with (16) becomes

$$g''+\beta_\kappa z_\kappa g = \frac{\alpha}{\kappa}\Theta_0' f, \tag{25}$$

where

$$z_\kappa = y+z_{\kappa 0},\qquad z_{\kappa 0} = (U_0^*-i\alpha\kappa)/U_0',\qquad \beta_\kappa = -i\alpha U_0'/\kappa_n. \tag{26}$$

Defining functions $G^{(1)}$ and $G^{(2)}$ as

$$G^{(1,2)}(z_\kappa) = \sqrt{z_\kappa}\,H_{1/3}^{(1,2)}(\tfrac{2}{3}\sqrt{\beta_\kappa}\,z_\kappa^{3/2}),$$

we obtain the following solutions of equation (25) for the upper and lower fluids, respectively,

$$\left.\begin{aligned} g_1 &= C_1 G_1^{(2)} + \frac{i\pi}{6}\frac{\alpha_1}{\kappa_1}\Theta_{10}' \times\left\{G_1^{(2)}\int_{z_{\kappa 10}}^{z_{\kappa 1}} G_1^{(1)} f_1\,\mathrm{d}\xi + G_1^{(1)}\int_{z_{\kappa 1}}^{\infty} G_1^{(2)} f_1\,\mathrm{d}\xi\right\},\\ g_2 &= C_2 G_2^{(1)} + \frac{i\pi}{6}\frac{\alpha_2}{\kappa_2}\Theta_{20}' \times\left\{G_2^{(2)}\int_{\hat{z}_{\kappa 2}}^{\infty} G_2^{(1)} f_2\,\mathrm{d}\xi + G_2^{(1)}\int_{\hat{z}_{\kappa 20}}^{\hat{z}_{\kappa 2}} G_2^{(2)} f_2\,\mathrm{d}\xi\right\}.\end{aligned}\right\} \tag{27}$$

With equations (20) and (22), equation (27) gives values of g and g' at the interface

$$\left.\begin{aligned} g_{n0} &= A_n G_{an0}+B_n G_{bn0}+C_n G_{cn0},\\ g_{n0}' &= A_n G_{an0}'+B_n G_{bn0}'+C_n G_{cn0}',\end{aligned}\right\}\ (n=1,2) \tag{28}$$

where G_{mn0} and G_{mn0}' are functions of $z_{\kappa n0}$ and given by equation (A3) in Appendix.

Replacing κ and Θ_0' by ε and W_0' in equation (27), we obtain in the same way values of h and h' at the interface

$$\left.\begin{aligned} h_0 &= A_1 H_{a0}+B_1 H_{b0}+DH_{d0},\\ h_0' &= A_1 H_{a0}'+B_1 H_{b0}'+DH_{d0}',\end{aligned}\right\} \tag{29}$$

where H_{m0} and H_{m0}' are functions of $z_{\varepsilon 0}$ and given by equation (A4) in Appendix.

Since we may consider the length $|\beta|^{-1/3}$ as a measure of the effective thickness of the disturbance field, the assumption that this thickness is small compared with the thickness of linear profile of the mean field can therefore be expressed as $|\beta|^{-1/3}\alpha = O(1)$ or less which means

$$\alpha_1\left(\frac{\alpha_1 U_{10}'}{\nu_1}\right)^{-1/3} < 1. \tag{30}$$

Thus, for sufficiently small wave numbers, taking account of $U_0^* = O(U_0)$, we may take $|\beta^{1/2} z_0^{3/2}| \ll 1$ which requires

$$\left(\frac{\alpha_1 U_{10}'}{\nu_1}\right)^{1/2}\left(\frac{U_0^*}{U_{10}'}\right)^{3/2} \ll 1. \tag{31}$$

For the present configuration of the flow system, the last condition could be always satisfied if α_1 and ν_1 have the relation of $\alpha_1 U_{10}'/\nu_1 = O(1)$ or less. We can now approximate τ_n as

$$\tau_n = \frac{2\cdot 3^{1/3}}{\Gamma(\frac{2}{3})}\frac{\beta_n^{1/3}}{\alpha_n}\exp\left\{-(-1)^n\frac{i\pi}{3}\right\}, \tag{32}$$

which means $|\tau_n| \gg 1$, and similarly G_{mn0}, G_{mn0}', H_{m0} and H_{m0}' as a power function of $\beta_{\kappa n}$ or β_ε as shown in Appendix (A5).

Substituting values of f, g and h at the interface (23), (28) and (29) with these approximations into the boundary condition (15), we can determine constants A_n, B_n, C_n and D to obtain the disturbance field at the interface, equations (A8) and (A9).

The disturbance normal velocity at $y = 0$, $\alpha_1 f_{10}$, is thus explicitly given by equation (A8) as

$$f_{10}\Bigg[\alpha_1 + e^{-i\pi/3}\Gamma_0\beta_{\kappa 1}^{1/3}\frac{\lambda_1}{L}\Bigg(i3^{-2/3}\Gamma(\tfrac{1}{3})\beta_{\kappa 1}^{1/3}\frac{\Theta'_{10}}{U'_{10}} + e^{i2\pi/3}\frac{\alpha_1}{H_t}\left\{1+e^{-i2\pi/3}\frac{\lambda_2}{\lambda_1}\left(\frac{\beta_{\kappa 2}}{\beta_{\kappa 1}}\right)^{1/3}\right\} \times\left\{H_p k_{a1} + \frac{1-W_0-3^{-1/3}\Gamma(\tfrac{2}{3})\beta_\varepsilon^{-1/3}W'_0}{V_{10}+e^{i\pi/3}\Gamma_0\varepsilon\beta_\varepsilon^{1/3}}\right\}\Bigg)\Bigg]$$
$$= iU_0^*\left\{\alpha_1 + i\,e^{i\pi/3}3^{-1/3}\Gamma(\tfrac{2}{3})\beta_{\kappa 2}^{2/3}\frac{\lambda_2}{L}\frac{\nu_1}{\nu_2}\frac{\Theta'_{20}}{U'_{20}}\right\} + e^{i\pi/3}\Gamma_0\beta_{\kappa 1}^{1/3}\frac{\lambda_1}{L}\Bigg(\left\{-\Theta'_{10}+e^{-i2\pi/3}\frac{\lambda_2}{\lambda_1}\frac{\nu_1}{\nu_2}\left(\frac{\beta_{\kappa 2}}{\beta_{\kappa 1}}\right)^{1/3}\Theta'_{20}\right\} + \frac{1}{H_t}\left\{1+e^{-i2\pi/3}\frac{\lambda_2}{\lambda_1}\left(\frac{\beta_{\kappa 2}}{\beta_{\kappa 1}}\right)^{1/3}\right\} \times\left\{-H_p g_1 + \frac{i\alpha_1(W_0-1)U_0^* + e^{i\pi/3}\Gamma_0\varepsilon\beta_\varepsilon^{1/3}W'_0}{V_{10}+e^{i\pi/3}\Gamma_0\varepsilon\beta_\varepsilon^{1/3}}\right\}\Bigg), \quad (33)$$

where $\Gamma_0 = 3^{1/3}\Gamma(\frac{2}{3})/\Gamma(\frac{1}{3})$. The right side of the above equation means the disturbance heat flux into the interface and the left is the disturbance amount of phase-change. Both sides physically consist of three terms, attributed to the disturbed convection field ($\sim\alpha_1$), the disturbed temperature field ($\sim\Theta'_0$) and the change in the interface temperature ($\sim H_t^{-1}$), respectively. The disturbance rate of phase-change $\tilde{v}_{10}/(V_{10}\delta_1) = \alpha_1(f_1 - iU_0^*)/V_{10}$ is graphically shown in Fig. 2. The order estimation of the terms of the above equation shows that the most predominant on the left side is the one attributed to the disturbance convection field ($\sim\alpha_1$) and on the right those due to the disturbed temperature field ($\sim\Theta'_0$) and due to the disturbed convection field ($\sim\alpha_1$), so that we can approximate the disturbance normal velocity as

$$\alpha_1 f_{10} = e^{i\pi/3}\Gamma_0\beta_{\kappa 1}^{1/3}\frac{\lambda_1}{L}(-\Theta'_{10})^* + i\alpha_1 U_0^*, \quad (34)$$

$$(-\Theta'_{10})^* \equiv -\Theta'_{10} + e^{-i2\pi/3}\frac{\lambda_2}{\lambda_1}\frac{\nu_1}{\nu_2}\left(\frac{\beta_{\kappa 2}}{\beta_{\kappa 1}}\right)^{1/3}\Theta'_{20},$$

or the disturbance rate of phase-change as

$$\tilde{v}_{10} = e^{i7\pi/6}\frac{\Gamma_0}{-L}\left(\frac{\alpha_1 U'_{10}}{\kappa_1}\right)^{1/3}\kappa_1(-\Theta'_{10})^*\delta_1, \quad (35)$$

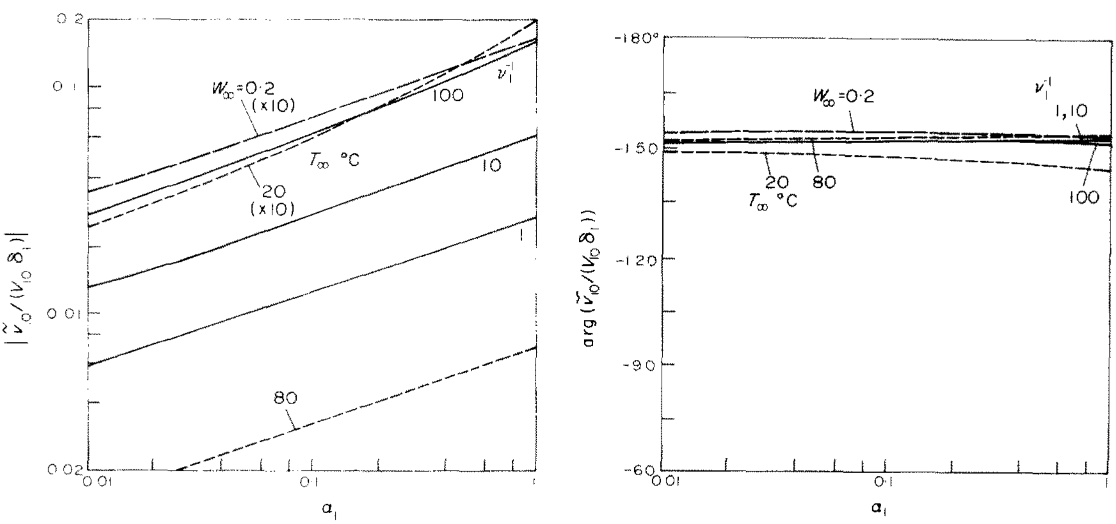
$$(-\Theta'_{10})^* = -\Theta'_{10} + e^{-i\pi/3}\left(\frac{\rho_1\kappa_1}{\rho_2\kappa_2}\right)^{1/3}\frac{\lambda_2}{\lambda_1}\frac{\nu_1}{\nu_2}\Theta'_{20},$$

where $U'_{20}/U'_{10} = \rho_1\nu_1/(\rho_2\nu_2)$ is used. The normal velocity at the interface is thus proportional to the 1/3rd power of the wave number with the phase lag of 150° relative to the interface, as shown in the figure.

The disturbance velocity gradient at the interface $\tilde{u}'_{10}/(U'_{10}\delta_1) = if''_{10}/U'_{10}$ given by equation (A8) is shown in Fig. 3. It is noted that $\tilde{u}'_{10}$ implies the disturbance friction coefficient at the interface c_f defined by

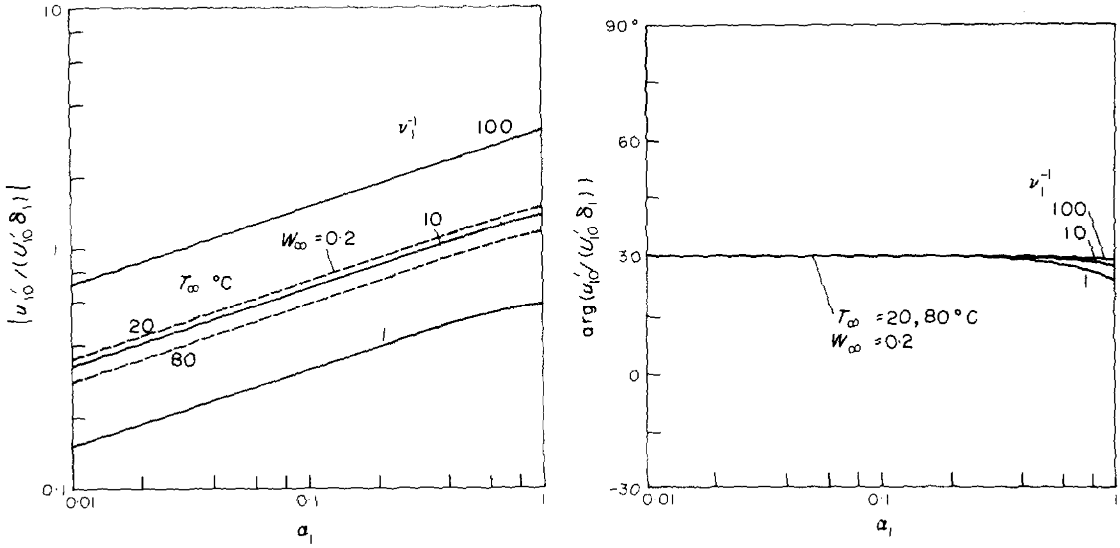
$$\rho_1\nu_1\left(\frac{\partial u_1}{\partial y_1}\right)_0 = c_f\rho_1(U_{1\infty} - U_{2\infty})^2,$$

of which dimensionless form is $c_f l_{r1}/\nu_1 = \tilde{u}'_{10}$. With equation (34) and the relation that $|\tau_1|, |\tau_2| \gg 1$, we



(a) Amplitude. (b) Phase angle.

FIG. 2. Disturbance rate of phase-change. Water–air, $T_{1\infty} = 100$°C, $c = 0$;—, $T_{2\infty} = 60$°C, $W_\infty = 0$;– – –, $\nu_1^{-1} = 10$, $W_\infty = 0$; ———, $\nu_1^{-1} = 10$, $T_{2\infty} = 60$°C: $V_{10}(T_{2\infty}$ °C, $W_\infty) = 0{\cdot}00123(20, 0), 0{\cdot}00989(60, 0), 0{\cdot}000472(60, 0{\cdot}2), 0{\cdot}0245(80, 0)$.



(a) Amplitude. (b) Phase angle.

FIG. 3. Disturbance friction coefficient. Water–air, $T_{1\infty} = 100$°C, $c = 0$; —, $T_{2\infty} = 60$°C, $W_\infty = 0$; ---, $\nu_1^{-1} = 10$, $W_\infty = 0$; ———, $\nu_1^{-1} = 10$, $T_{2\infty} = 60$°C: $U'_{10}(T_{2\infty}$°C, $W_\infty) = 0{\cdot}323(20, 0), 0{\cdot}263(60, 0), 0{\cdot}329(60, 0{\cdot}2), 0{\cdot}168(80, 0)$.

obtain the following approximate expression for f''_{10},

$$f''_{10} = e^{i5\pi/3}\Gamma_1\alpha_1\left(\frac{\alpha_1 U'_{10}}{\nu_1}\right)^{1/3} U_0^* + e^{i5\pi/3}\Gamma_1\left(\frac{\alpha_1 U'_{10}}{\nu_1}\right)^{1/3}\left(U'_{10} - \frac{\nu_1}{\nu_2}U'_{20}\right) + e^{i\pi/3}\frac{\Gamma_2}{-L}\left(\frac{\alpha_1 U'_{10}}{\nu_1}\right)^{2/3}(\nu_1\kappa_1^2)^{1/3}(-\Theta'_{10})^* + e^{i5\pi/6}\,2U_0^*\alpha_1^2\frac{\nu_2}{\nu_1}\left(\frac{U'_{10}}{U'_{20}}\right)^{1/3}, \quad (36)$$

where $\Gamma_1 = 3^{1/3}/\Gamma(\frac{2}{3})$ and $\Gamma_2 = 3^{2/3}/\Gamma(\frac{1}{3})$. The right side of the above equation comprises two contributions from the disturbance fields of the modified mean field of velocity [the first ($\sim\alpha^{4/3}$) and second ($\sim\alpha^{1/3}$) terms] and of the disturbed velocity field due to the phase-change [the third ($\sim\alpha^{2/3}$) and the fourth ($\sim\alpha^2$)]. The latter contribution, that is, the phase-change at the interface has less effect upon the friction coefficient u'_{10}, which is then largely dominated by the second term, being proportional to $(\alpha_1/\nu_1)^{1/3}$ and 30° in advance of the interface.

The disturbance pressure $p_{10}/\delta_1 = k_{10}$ is given by equation (13) and shown in Fig. 4, being proportional to $(\alpha_1/\nu_1)^{-1/3}$ and $(\alpha_1/\nu_1)^{2/3}$ at smaller and larger wave numbers, respectively. In the analytical expression of k by equation (13), the most predominant is $\nu f'''$;

$$\alpha_1 k_{10} \approx \nu_1 f'''_{10}.$$

Since equation (20) gives f'''_{10} approximately as

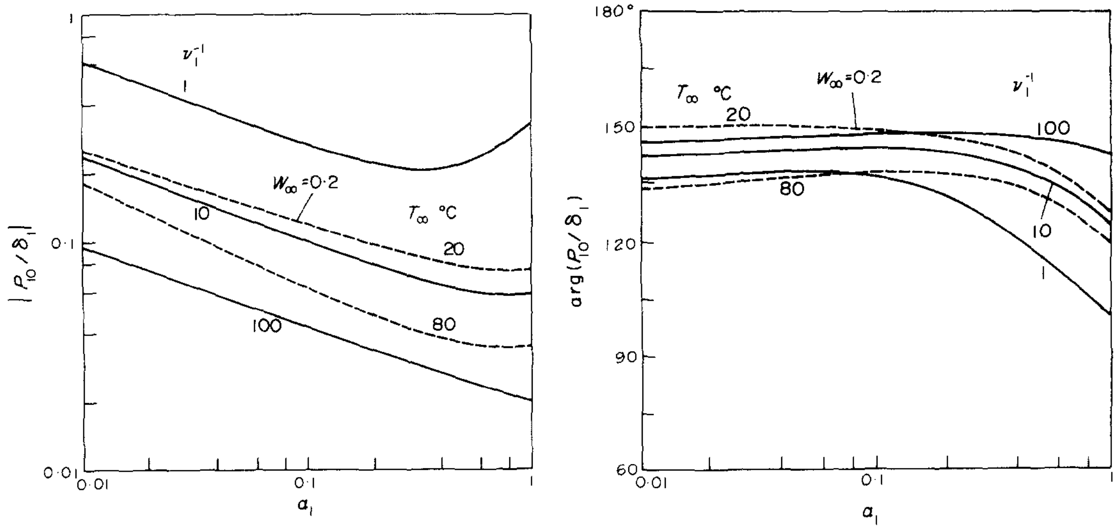
$$f'''_{10} = e^{i\pi/3}\Gamma_2\alpha_1\left(f_{10} + \frac{f'_{10}}{\alpha_1}\right)\left(\frac{\alpha_1 U'_{10}}{\nu_1}\right)^{2/3},$$

with equation (34) and (42), k_{10} is then given by

$$k_{10} = e^{i\pi/3}\Gamma_2\left\{e^{i7\pi/6}\frac{\Gamma_0}{-L}(U'_{10})^{1/3}\left(\frac{\kappa_1}{\alpha_1}\right)^{2/3}(-\Theta'_{10})^* + iU_0 + \frac{iU'_{10}}{\alpha_1}\right\}\nu_1\left(\frac{\alpha_1 U'_{10}}{\nu_1}\right)^{2/3}, \quad (37)$$

of which the third and second terms become more effective at smaller and larger wave numbers, being proportional to $\alpha_1^{-1/3}$ and $\alpha_1^{2/3}$, respectively, with the phase advance of 150°.

An interesting feature of equations (36) and (37) is the phase relation between the stress and the wave at the interface. To first approximation, shearing stress ($\sim u'_{10}$) is approximately 30° in advance of the wave, while the phase of the normal stress is about 150° in advance. These phase relations are accordant with the Benjamin's result of linear or boundary-layer profile model, which may be interpreted as a kind of Jefffereys' "sheltering" effect that the stresses are distributed as if the leeward slopes of the wavy interface were sheltered and a wake were formed behind each wave crest. The effect of the phase-change upon the stresses at the interface is expressed by the terms of $(-\Theta'_{10})^*$ which becomes relatively predominant at larger wave numbers or for the case of higher vapor concentrations



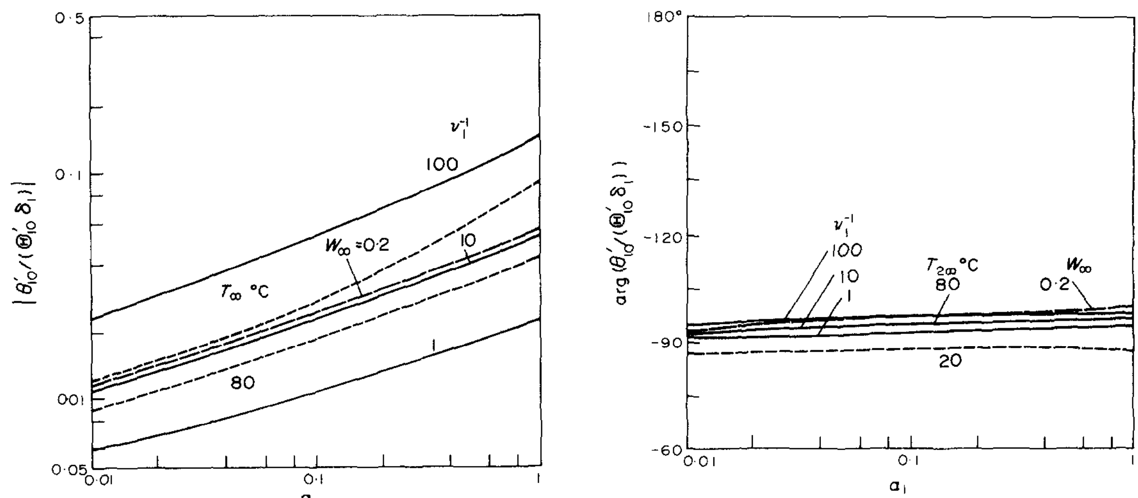
(a) Amplitude. (b) Phase angle.

FIG. 4. Disturbance pressure at the interface. Water-air, $T_{1\infty} = 100°C$, $c = 0$; —, $T_{2\infty} = 60°C$, $W_\infty = 0$; ---, $\nu_1^{-1} = 10$, $W_\infty = 0$; ———, $\nu_1^{-1} = 10$, $T_{2\infty} = 60°C$.

at infinity (condensation taking place at the interface). They have the phase relation of 150° for u'_{10} and $-90°$ for k_{10} in advance, respectively, and act to weaken the "sheltering" effect in the phase relation. In the evaporation case, the "sheltering" effect may be also reduced in amplitude by the decrease of U'_{10} and $-\Theta'_{10}$.

The disturbance temperature gradient at the interface $\theta'_{10}/(\Theta'_{10}\delta_1) = g'_{10}/\Theta'_{10}$ given by equation (A9) is shown in Fig. 5, being proportional to $\alpha_1^{0\sim 2/3}$. Here, $|-\theta'_{10}|$ implies the amplitude of the disturbance heat-transfer coefficient at the interface c_h defined as

$$\lambda_1\left(\frac{\partial T_1}{\partial y_1}\right)_0 = c_h(T_{1\infty} - T_{2\infty}),$$



(a) Amplitude. (b) Phase angle.

FIG. 5. Disturbance heat-transfer coefficient. Water-air, $T_{1\infty} = 100°C$, $c = 0$; —, $T_{2\infty} = 60°C$, $W_\infty = 0$; ---, $\nu_1^{-1} = 10$, $W_\infty = 0$; ———, $\nu_1^{-1} = 10$, $T_{2\infty} = 60°C$: $-\Theta'_{10}(T_{2\infty}\,°C, W_\infty) = 0{\cdot}288(20, 0), 0{\cdot}243(60, 0), 0{\cdot}292(60, 0{\cdot}2), 0{\cdot}169(80, 0)$.

of which non-dimensional form is $c_h l_{r1}/\lambda_1 = -\theta'_{10}$. The approximate expression for g'_{10} is

$$g'_{10} = i\,e^{i2\pi/3}\,\Gamma_0 (U'_{10})^{1/3}\left(\frac{\alpha_1}{\kappa_1}\right)^{1/3} \cdot\Bigg[(f_{10}-iU_0^*)(-\Theta'_{10})\left\{1+e^{i2\pi/3}\,\Gamma_3 (U'_{10})^{-2/3}\left(\frac{\alpha_1}{\kappa_1}\right)^{1/3}\right\}+e^{i\pi/6}(f_{10}-iU_0^*)\frac{-L}{\Gamma_0}\frac{\lambda_1}{\lambda_2}\left(\frac{\nu_2}{\nu_1}\frac{\kappa_1}{\kappa_2}U'_{20}\right)^{-1/3}\left(\frac{\alpha_1}{\kappa_1}\right)^{2/3}$$
$$+iU_0^*\left((-\Theta'_{10})\left\{1+e^{i2\pi/3}\,\Gamma_3 (U'_{10})^{-2/3}\left(\frac{\alpha_1}{\kappa_1}\right)^{1/3}\right\}+\Theta'_{20}\left\{1+e^{i\pi/3}\,\Gamma_3\frac{\nu_2}{\nu_1}\left(\frac{\nu_2\kappa_1}{\nu_1\kappa_2}\right)^{1/3}(U'_{20})^{-2/3}\left(\frac{\alpha_1}{\kappa_1}\right)^{1/3}\right\}\right)\Bigg], \quad (38)$$

where $\Gamma_3 = 3^{-7/6}/\Gamma(\frac{2}{3})$, which comprises the effects of the disturbed temperature field due to the disturbed flow (the first and third terms $\sim \Theta'_0$) and the disturbance amount of heat for phase-change (the second $\sim L$). The predominant terms of them yield

$$\frac{\theta'_{10}}{\Theta'_{10}\delta_1} = \frac{\Gamma(\frac{2}{3})\pi}{\sqrt{3}\,\Gamma(\frac{1}{3})\Gamma(\frac{1}{3})}\frac{1}{-L}(-\Theta'_{10})^* + \frac{3^{1/3}\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\left\{e^{-i\pi/2}\frac{\lambda_1}{\lambda_2}\left(\frac{\rho_1\kappa_1}{\rho_2\kappa_2}\right)^{-1/3}\frac{(-\Theta_{10})^*}{-\Theta_{10}} + e^{i2\pi/3}\,U_0^*\left(1-\frac{\Theta'_{20}}{\Theta'_{10}}\right)\right\}\left(\frac{\alpha_1 U'_{10}}{\kappa_1}\right)^{1/3}, \quad (39)$$

which shows the proportionality to $(\alpha_1/\kappa_1)^{1/3}$ and the phase lag of 90° in the case of intense evaporation.

Comparing the predominant terms of equations (36) and (39) gives

$$\left|\frac{\theta'_{10}}{\Theta'_{10}}\right| \Big/ \left|\frac{u'_{10}}{U'_{10}}\right| \approx \frac{\lambda_1}{\lambda_2}\left(\frac{\rho_2\kappa_2}{\rho_1\kappa_1}\right)^{1/3}\left(\frac{\nu_1}{\kappa_1}\right)^{1/3}, \quad (40)$$

of which the order of magnitude is about 0·1, being contrasted with the steady value $|\Theta'_{10}/U'_{10}| \approx 1$. This indicates that the waviness of the interface causes disturbance to the heat-transfer coefficient by one order of magnitude less than to the skin-friction. Of the temperature field, the wavy disturbance at the interface is almost absorbed into the liquid because of its high heat conductivity and the heat flux required for the disturbance rate of phase-change at the interface is supplied mostly by the heat conduction through the liquid layer.

The disturbance coefficient of heat transfer of the liquid side is then given by

$$g'_{20} \approx e^{i7\pi/6}\frac{3^{1/3}\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}\frac{\lambda_1}{\lambda_2}\left(\frac{\alpha_1 U'_{10}}{\kappa_1}\right)^{1/3}(-\Theta'_{10}), \quad (41)$$

of which the ratio to g'_{10} is $(\rho_1\kappa_1/\rho_2\kappa_2)^{1/3} = 0{\cdot}1$–$1$. On the other hand, the ratio of f''_{20} to f''_{10} is $\rho_1\nu_1/\rho_2\nu_2 = 10^{-2}$–$10^{-3}$.

The disturbance u-velocity and temperature at the interface given by equations (A8) and (A9) are approximated as

$$if'_{10} \approx -U'_{10}\left(1-\frac{\rho_1\nu_1^2}{\rho_2\nu_2^2}\right), \quad (42)$$

$$g_{10} \approx e^{-i2\pi/3}\frac{\lambda_1}{\lambda_2}\left(\frac{\rho_1\kappa_1}{\rho_2\kappa_2}\right)^{-1/3}. \quad (43)$$

To first approximation, both disturbances are relevant to neither the wave number nor the Reynolds number. These equations show that the overall disturbance u-velocity at the interface $\tilde{u}_{10}$, which is the sum of $if'_{10}\delta_1$ and $U'_{10}\delta_1$, is nearly equal to zero; that is, the overall u-velocity at the interface remains undisturbed. Since $|g_{10}| \ll |\Theta'_{10}|$, the overall disturbance temperature at the interface is then roughly $\tilde{\theta}_{10} \approx \Theta'_{10}\delta_1$.

CONCLUSION

Laminar boundary-layer flows of gas and liquid having a phase-changing interface are perturbed with a small wavy disturbance to examine the aspect of their disturbed fields and the effect of the phase-change upon the hydrodynamic instability of the system. To obtain quantitative knowledge of the wavy disturbed field, the problem is treated analytically with an approximation of linear profiles for the base field.

Corresponding to the disturbance elevation of the interface $\delta_0 e^{i\alpha x}$, the disturbance u-velocity gradient at the interface, that is, the skin-friction is

$$\frac{u'_{10}}{U'_{10}} = \left\{1{\cdot}066\left(1-\frac{\rho_1\nu_1^2}{\rho_2\nu_2^2}\right)\left(\frac{\alpha_1 U'_{10}}{\nu_1}\right)^{1/3} e^{i\pi/6} + 0{\cdot}776\frac{(\nu_1\kappa_1^2)^{1/3}}{-L}\left(\frac{\alpha_1 U'_{10}}{\nu_1}\right)^{2/3}\frac{(-\Theta'_{10})^*}{U'_{10}}e^{i5\pi/6}\right\}\delta_{10}e^{i\alpha x}.$$

The disturbance pressure acting on the interface is given by

$$p_{10} = \left\{0{\cdot}565\frac{1}{-L}\left(\frac{\kappa_1}{\nu_1}\right)^{2/3}U'_{10}(-\Theta'_{10})^* e^{-i\pi/2} + 0{\cdot}776\left(U_0+\frac{U'_{10}}{\alpha_1}\right)\left(\frac{\alpha_1 U'_{10}}{\nu_1}\right)^{2/3}e^{i5\pi/6}\right\}\nu_1\delta_{10}e^{i\alpha x}.$$

To first approximation, the phase relation of the shearing stress (+30°) and the normal stress (+150°) relative to the wave at the interface is accordant with the Benjamin's result for the case of isothermal flows, showing the "sheltering" effect. The phase-change of evaporation at the interface acts to weaken such a "sheltering" effect both in amplitude and in phase relation, especially at larger wave numbers.

The disturbance rate of phase-change at the interface is approximately estimated as

$$\tilde{v}_{10} = 0{\cdot}728\,\frac{\lambda_1}{-L}\left(\frac{\alpha_1 U'_{10}}{\kappa_1}\right)^{1/3}(-\Theta'_{10})^*\delta_{10}\,e^{i(\alpha x - 5\pi/6)},$$

which is proportional to $(\alpha_1/\kappa_1)^{1/3}$ with the phase lag of 150°. The temperature gradient at the interface, that is, the heat-transfer coefficient is

$$\frac{\theta'_{10}}{\Theta'_{10}} = \left[0{\cdot}342\,\frac{(-\Theta'_{10})^*}{-L} + 0{\cdot}728\left(\frac{\alpha_1 U'_{10}}{\kappa_1}\right)^{1/3} \cdot\left\{\frac{\lambda_1}{\lambda_2}\left(\frac{\rho_1\kappa_1}{\rho_2\kappa_2}\right)^{-1/3}\frac{(-\Theta'_{10})^*}{-\Theta'_{10}}\,e^{-i\pi/2} + U_0^*\left(1-\frac{\Theta'_{20}}{\Theta'_{10}}\right)e^{i2\pi/3}\right\}\right]\delta_{10}\,e^{i\alpha x},$$

which, for intense evaporation, is proportional to $(\alpha_1/\kappa_1)^{1/3}$ with the phase lag of 90°.

Because of the high heat conductivity of the liquid, the heat flux required for the disturbance rate of phase-change at the interface is supplied mainly by heat conduction through the liquid layer, so that the wavy disturbance has less influence upon the heat-transfer coefficient than upon the skin-friction. The ratio of the former to the latter is about

$$\lambda_1/\lambda_2(\rho_2\kappa_2/\rho_1\kappa_1)^{1/3} \approx 0{\cdot}1.$$

The disturbance u-velocity and temperature at the interface are relevant to neither the wave number nor the Reynolds number and approximately given by

$$u_{10} \approx U'_{10}\left(1-\frac{\rho_1\nu_1^2}{\rho_2\nu_2^2}\right)\delta_{10}\,e^{i(\alpha x+\pi)},$$

$$\theta_{10} \approx \frac{\lambda_1}{\lambda_2}\left(\frac{\rho_1\kappa_1}{\rho_2\kappa_2}\right)^{-1/3}(-\Theta'_{10})\delta_{10}\,e^{i(\alpha x + 2\pi/3)},$$

which have the phase advance of 180° and 120°, respectively; that is, the overall disturbance u-velocity and temperature at the interface can be approximated as $\tilde{u}_{10} \approx 0$ and $\tilde{\theta}_{10} \approx \Theta'_{10}\delta_{10}\,e^{i\alpha x}$, respectively.

The assumptions which form the basis for the approximations introduced in the present theory may restrict the validity of the obtained result within the range of parametrical quantities α_1 and ν_1 that

$$\alpha_1\left(\frac{\alpha_1 U'_{10}}{\nu_1}\right)^{-1/3} < 1$$

and

$$\left(\frac{\alpha_1 U'_{10}}{\nu_1}\right)^{1/2}\left(\frac{U_0^*}{U'_{10}}\right)^{3/2} \ll 1.$$

Since the latter is satisfied for the present flow-configuration if $\alpha_1 U'_{10}/\nu_1 \lesssim 1$, these restrictions can be arranged to give

$$\alpha_1 < \left(\frac{\alpha_1 U'_{10}}{\nu_1}\right)^{1/3} < 1.$$

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APPENDIX

Disturbance Field at the Interface

The boundary conditions at the interface (15) with $\delta_0 = 1$ are expressed in a matrix form

$$\begin{pmatrix} a_{u1} & b_{u1} & a_{u2} & b_{u2} & 0 & 0 & 0 & e_u \\ a_{s1} & b_{s1} & a_{s2} & b_{s2} & 0 & 0 & 0 & e_s \\ a_{c1} & b_{c1} & a_{c2} & b_{c2} & 0 & 0 & 0 & e_c \\ a_{t1} & b_{t1} & a_{t2} & b_{t2} & c_{t1} & c_{t2} & 0 & e_t \\ a_{q1} & b_{q1} & a_{q2} & b_{q2} & c_{q1} & c_{q2} & 0 & e_q \\ a_{w1} & b_{w1} & 0 & 0 & 0 & 0 & d_w & e_w \\ a_{e1} & b_{e1} & 0 & 0 & c_{e1} & 0 & d_e & e_e \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ C_1 \\ C_2 \\ D \\ 1 \end{pmatrix} = 0 \qquad \text{(A1)}$$

where $a_{mn}, b_{mn}, c_{mn}, d_m$ and $e_m (n = 1, 2)$ are from equation (10) as follows;

$$a_{un} = -(-1)^n (f'_{an})_0 \quad b_{un} = (a_{un})_{a\to b}$$
$$e_u = -i(U'_{10} - U'_{20}\nu_1/\nu_2)$$
$$a_{sn} = -(-1)^n \rho_n \nu_n (f'''_{an} + \alpha_n^2 f_{an})_0 \quad b_{sn} = (a_{sn})_{a\to b}$$
$$e_s = -i\nu_1(\rho_1 U''_{10} - \rho_2 U''_{20})$$
$$a_{cn} = -(-1)^n \rho_n \nu_n (f_{an})_0 \quad b_{cn} = (a_{cn})_{a\to b}$$
$$e_c = -i(\rho_1 - \rho_2)\nu_1 U_0^*$$
$$a_{tn} = (g_{an})_0 \quad b_{tn} = (g_{bn})_0 \quad c_{tn} = (g_{cn})_0$$
$$e_t = \Theta'_{10} + \Theta'_{20}\nu_1/\nu_2$$
$$a_{q1} = \alpha_1 (f_{a1})_0 - \lambda_1/L(g'_{a1})_0 \quad b_{q1} = (a_{q1})_{a\to b}$$
$$a_{q2} = -\lambda_2/L(g'_{a2})_0 \quad b_{q2} = (a_{q2})_{a\to b} \quad c_{qn} = -\lambda_n/L(g'_{cn})_0$$
$$e_q = -i\alpha_1 U_0^* - (\lambda_1 \Theta''_{10} + \nu_1/\nu_2 \,.\, \lambda_2 \Theta''_{20})/L$$
$$a_{w1} = \{\alpha_1(W-1)f_{a1} + V_1 h_{a1} - \varepsilon h'_{a1}\}_0 \quad b_{w1} = (a_{w1})_{a\to b}$$
$$d_w = (V_1 h_d - \varepsilon h'_d)_0 \quad e_w = \{-i\alpha_1(W-1)U^* + V_1 W' - \varepsilon W''\}_0$$
$$a_{e1} = (h_a - H_t g_{a1} + H_p k_{a1})_0 \quad b_{e1} = (a_{e1})_{a\to b}$$
$$c_{a1} = -H_t(g_{c1})_0 \quad d_e = (h_d)_0 \quad e_e = -H_p \boldsymbol{g}_1 + W'_0 - H_t \Theta'_{10}$$

where $(\ )_{a\to b}$ means that all values of the type ϕ_{an} included in ( ) are replaced by the corresponding values ϕ_{bn}.

Eliminating C_1, C_2 and D in equation (A1) becomes

$$\begin{bmatrix} a_{u1} & b_{u1} & a_{u2} & b_{u2} & e_u \\ a_{s1} & b_{s1} & a_{s2} & b_{s2} & e_s \\ a_{c1} & b_{c1} & a_{c2} & b_{c2} & e_c \\ a_{q1}^* & b_{q1}^* & a_{q2}^* & b_{q2}^* & e_q^* \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ 1 \end{bmatrix} = 0 \tag{A2}$$

where

$$a_{q1}^* = \alpha_1 (f_{a1})_0 + a_{q1}^{**}$$
$$a_{q1}^{**} \equiv \frac{\lambda_1}{L}\left(\frac{g_{a1}}{g_{c1}} g'_{c1} - g'_{a1}\right)_0 + \frac{\lambda_1}{LH_t}\left(\frac{\lambda_2}{\lambda_1}\frac{g'_{c2}}{g_{c2}} - \frac{g'_{c1}}{g_{c1}}\right)_0 \left\{H_p k_{a1} + \left[(1-W)\alpha_1 f_{a1} + \varepsilon h_{a1}\left(\frac{h'_{a1}}{h_{a1}} - \frac{h'_d}{h_d}\right)\right]_0 \times \left(V_1 - \varepsilon\frac{h'_d}{h_d}\right)_0^{-1}\right\}$$
$$b_{q1}^* = \alpha_1 (f_{b1})_0 + b_{q1}^{**} \qquad b_{q1}^{**} \equiv (a_{q1}^{**})_{a\to b}$$
$$a_{q2}^* = \frac{\lambda_2}{L}\left(\frac{g_{a2}}{g_{c2}} g'_{c2} - g'_{a2}\right)_0 \quad b_{q2}^* \equiv (a_{q2}^*)_{a\to b}$$
$$e_q^* = -i\alpha_1 U_0^* e_q^{**}$$
$$e_q^{**} \equiv \frac{\lambda_1}{L}\left(\frac{g'_{c1}}{g_{c1}}\Theta'_{10} - \Theta''_{10}\right) + \frac{\nu_1}{\nu_2}\frac{\lambda_2}{L}\left(\frac{g'_{c2}}{g_{c2}}\Theta'_{20} - \Theta''_{20}\right) + \frac{\lambda_1}{LH_t}\left(\frac{\lambda_2}{\lambda_1}\frac{g'_{c2}}{g_{c2}} - \frac{g'_{c1}}{g_{c1}}\right)_0 \left\{-H_p \boldsymbol{g}_1 + \left(V_1 - \varepsilon\frac{h'_d}{h_d}\right)_0^{-1} . \left[i(W-1)\alpha_1 U^* + \varepsilon W'\left(\frac{W''}{W'} - \frac{h'_d}{h_d}\right)\right]_0\right\}.$$

Values of f, g and h at the interface are given by equations (23), (28) and (29), where G_{mn0}, G'_{mn0}, H_{m0} and H'_{m0} are

$$\left.\begin{aligned} &G_{mn0} = G_{mn}G_n^{(n)}(z_{\kappa n0}) \quad G'_{mn0} = G_{mn}G_n^{(n)'}(z_{\kappa n0}) \qquad (m = a, b) \\ &G_{m1} = \frac{i\pi}{6}\frac{\alpha_1}{\kappa_1}\Theta'_{10}\int_{z_{\kappa 10}}^{\infty} G_1^{(1)} F_{m1}\,d\xi \\ &G_{m2} = \frac{i\pi}{6}\frac{\alpha_2}{\kappa_2}\Theta'_{20}\int_{\hat{z}_{\kappa 20}}^{\infty} G_2^{(2)} F_{m2}\,d\xi \\ &G_{c10} = G_1^{(2)}(z_{\kappa 10}) \quad G'_{c10} = G_1^{(2)'}(z_{\kappa 10}) \\ &G_{c20} = G_2^{(1)}(z_{\kappa 20}) \quad G'_{c20} = G_2^{(1)'}(z_{\kappa 20}) \end{aligned}\right\} \tag{A3}$$

$$\left.\begin{aligned} &H_{m0} = H_m H_\varepsilon^{(1)}(z_{\varepsilon 10}) \quad H'_{m0} = H_m H_\varepsilon^{(1)'}(z_{\varepsilon 10}) \\ &H_{d0} = H_\varepsilon^{(2)}(z_{\varepsilon 10}) \quad H'_{d0} = H_\varepsilon^{(2)'}(z_{\varepsilon 10}) \\ &H_m = \frac{i\pi}{6}\frac{\alpha_1}{\varepsilon} W'_0 \int_{z_{\varepsilon 10}}^{\infty} H_\varepsilon^{(1)} F_{m1}\,d\xi \\ &H_\varepsilon^{(1)} = \sqrt{z} H_{1/3}^{(1)}(\tfrac{2}{3}\sqrt{\beta_\varepsilon} z^{3/2}) \end{aligned}\right\} \tag{A4}$$

where F_{an} and F_{bn} are the coefficients of A_n and B_n on the right of equations (20) and (22), respectively. When $|\beta^{1/2} z_0^{3/2}| \ll 1$, we can approximate these values as follows;

$$\left.\begin{aligned} &G_{an} = G_{bn} = \frac{i\pi}{6}\frac{2}{\sqrt{3}}\frac{\alpha_n}{\kappa_n}\Theta'_{n0}\beta_{\kappa n}^{-1/2} e\left(n, \frac{i\pi}{6}\right) \\ &G_n^{(m)}(z_{\kappa n0}) = \frac{2}{\sqrt{3}}\frac{3^{1/3}}{\Gamma(\frac{2}{3})}\beta_{\kappa n}^{-1/6} e\left(m, -\frac{i\pi}{2}\right) \\ &G_n^{(m)'}(z_{\kappa n0}) = \frac{2}{\sqrt{3}}\frac{3^{2/3}}{\Gamma(\frac{1}{3})}\beta_{\kappa n}^{1/6} e\left(m, \frac{i\pi}{6}\right), \end{aligned}\right\} \tag{A5}$$
$$e(n, x) \equiv \exp\{-(-1)^n x\}$$

which hold for H and H' by replacing κ and Θ'_0 by ε and W'_0, respectively.

Substituting values of f, g and h at the interface given by equations (23), (28) and (29) with (A3), (A4), (A5) and (32) into the boundary conditions (A2) yields

$$\begin{bmatrix} -\alpha_1 & \alpha_1 & -\alpha_2 & \alpha_2 & -i\left(U'_{10} - U'_{20}\dfrac{\nu_1}{\nu_2}\right) \\ \dfrac{\rho_1}{\rho_2}\left(\dfrac{\nu_1}{\nu_2}\right)^3 & \dfrac{\rho_1}{\rho_2}\left(\dfrac{\nu_1}{\nu_2}\right)^3\left(1 - \dfrac{\tau_1}{2}\right) & -1 & -\left(1 - \dfrac{\tau_2}{2}\right) & 0 \\ \dfrac{\rho_1}{\rho_2} & \dfrac{\rho_1}{\rho_2} & -\dfrac{\nu_2}{\nu_1} & -\dfrac{\nu_2}{\nu_1} & -i\left(\dfrac{\rho_1}{\rho_2} - 1\right)U_0^* \\ \alpha_1 + a_1^* & \alpha_2 + b_1^* & a_2^* & b_2^* & -i\alpha_1 U_0^* + e^* \end{bmatrix} \begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ 1 \end{bmatrix} = 0 \tag{A6}$$

where

$$a_1^* = b_1^* = e^{-i\pi/3}\, 3^{1/3} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \frac{\lambda_1}{L} \beta_{\kappa 1}^{1/3} \left[i3^{-2/3}\Gamma(\tfrac{1}{3})\beta_{\kappa 1}^{1/3} \frac{\Theta'_{10}}{U'_{10}} + e^{i2\pi/3} \frac{\alpha_1}{H_t} \left\{ 1 + e^{-i2\pi/3} \left(\frac{\beta_{\kappa 2}}{\beta_{\kappa 1}} \right)^{1/3} \frac{\lambda_2}{\lambda_1} \right\} \frac{1 - W_0 - 3^{-1/3}\Gamma(\frac{2}{3})\beta_\varepsilon^{-1/3} W'_0}{V_{10} + e^{i\pi/3}\, 3^{1/3}\Gamma(\frac{2}{3})/\Gamma(\frac{1}{3})\varepsilon\beta_\varepsilon^{1/3}} \right]$$

$$a_2^* = b_2^* = -i\, e^{i\pi/3}\, 3^{-1/3}\Gamma(\tfrac{2}{3}) \frac{\lambda_2}{L} \beta_{\kappa 2}^{2/3} \frac{\Theta'_{20}}{U'_{20}}$$

$$e^* = -e^{i\pi/3}\, 3^{1/3} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \frac{\lambda_1}{L} \beta_{\kappa 1}^{1/3} \times \left[\left\{ -\Theta'_{10} + e^{-i2\pi/3} \left(\frac{\beta_{\kappa 2}}{\beta_{\kappa 1}} \right)^{1/3} \frac{\nu_1}{\nu_2} \frac{\lambda_2}{\lambda_1} \Theta'_{20} \right\} + \frac{1}{H_t} \left\{ 1 + e^{-i2\pi/3} \left(\frac{\beta_{\kappa 2}}{\beta_{\kappa 1}} \right)^{1/3} \frac{\lambda_2}{\lambda_1} \right\} \frac{i\alpha_1 (W_0 - 1) U_0^* + e^{i\pi/3}\, 3^{1/3}\Gamma(\frac{2}{3})/\Gamma(\frac{1}{3})\varepsilon\beta_\varepsilon^{1/3} W'_0}{V_{10} + e^{i\pi/3}\, 3^{1/3}\Gamma(\frac{2}{3})/\Gamma(\frac{1}{3})\varepsilon\beta_\varepsilon^{1/3}} \right].$$

Equation (A6) gives constants A_n and B_n

$$\left.\begin{aligned} \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} &= \frac{1}{2} \left\{ \mp \frac{i}{\alpha_1} \left(U'_{10} - \frac{\nu_1}{\nu_2} U'_{20} \right) \pm i \frac{4 - \tau_2}{\tau_2} U_0^* - i \frac{\nu_1}{\nu_2} \frac{a_2^*}{\alpha_1 + a_1^*} U_0^* - \frac{-i\alpha_1 U_0^* + e^*}{\alpha_1 + a_1^*} \right\} \\ \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} &= i \left(\frac{1}{2} \mp \frac{4 - \tau_2}{2\tau_2} \right) \frac{\nu_1}{\nu_2} U_0^* \end{aligned}\right\} \quad \text{(A7)}$$

where $\rho_1/\rho_2 \ll 1$ is assumed.

With these constants, equations (23) give the disturbance flow field at the interface;

$$\left.\begin{aligned} f_{10} &= A_1 + B_1 = \frac{1}{\alpha_1 + a_1^*} \left\{ i\alpha_1 \left(1 - \frac{a_2^*}{\alpha_2} \right) U_0^* - e^* \right\} \\ f_{20} &= A_2 + B_2 = i \frac{\nu_1}{\nu_2} U_0^* \\ f'_{10} &= \alpha_1(-A_1 + B_1) \\ &= i \left(U'_{10} - \frac{\nu_1}{\nu_2} U'_{20} \right) - i \frac{4 - \tau_2}{\tau_2} U_0^* \\ f''_{10} &= \alpha_1^2 (A_1 + B_1) - \alpha_1^2 \tau_1 B_1 \\ &= \alpha_1^2 \left\{ f_{10} - \frac{\tau_1}{2} \left(f_{10} + \frac{f'_{10}}{2} \right) \right\}. \end{aligned}\right\} \quad \text{(A8)}$$

By eliminating C_2 with equations $(A1)_t$ and $(A1)_q$ to obtain C_1, equation (28) gives the disturbance thermal field at the interface;

$$\left.\begin{aligned} g_{10} &= \left(\frac{\lambda_2}{\lambda_1} \frac{g_{c1}}{g'_{c1}} \frac{g'_{c2}}{g_{c2}} - 1 \right)_0^{-1} \left[-\frac{\lambda_2}{\lambda_1} \frac{g'_{c2}}{g_{c2}} \left\{ \frac{\lambda_1}{\lambda_2} \frac{g_{c2}}{g'_{c2}} g_{a1} \left(1 - \frac{g_{c1}}{g'_{c1}} \frac{g'_{a1}}{g_{a1}} \right) (A_1 + B_1) + \frac{g_{c1}}{g'_{c1}} g_{a2} \left(1 - \frac{g_{c2}}{g'_{c2}} \frac{g'_{a2}}{g_{a2}} \right) (A_2 + B_2) + \frac{g_{c1}}{g'_{c1}} e_t \right\} - \frac{L}{\lambda_1} \frac{g_{c1}}{g'_{c1}} \{ e_q + \alpha_1 (A_1 + B_1) \} \right]_0 \\ g'_{10} &= \left(\frac{\lambda_2}{\lambda_1} \frac{g_{c1}}{g'_{c1}} \frac{g'_{c2}}{g_{c2}} - 1 \right)_0^{-1} \left[-\frac{\lambda_2}{\lambda_1} \frac{g'_{c2}}{g_{c2}} \left\{ g_{a1} \left(1 - \frac{g_{c1}}{g'_{c1}} \frac{g'_{a1}}{g_{a1}} \right) (A_1 + B_1) + g_{a2} \left(1 - \frac{g_{c2}}{g'_{c2}} \frac{g'_{a2}}{g_{a2}} \right) (A_2 + B_2) + e_t \right\} - \frac{L}{\lambda_1} \{ e_q + \alpha_1 (A_1 + B_1) \} \right]_0 \end{aligned}\right\} \quad \text{(A9)}$$

where, in virtue of equation (A5), we can use the following approximation

$$\frac{g'_{an0}}{g_{an0}} = i3^{1/3} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \beta_{\kappa n}^{1/3} e\left(n, \frac{i\pi}{6} \right)$$

$$\frac{g'_{cn0}}{g_{cn0}} = -i3^{1/3} \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})} \beta_{\kappa n}^{1/3} e\left(n, -\frac{i\pi}{6} \right)$$

$$g_{an0} = i \frac{2\pi}{9} \frac{3^{1/3}}{\Gamma(\frac{2}{3})} \frac{\alpha_n}{\kappa_n} \beta_{\kappa n}^{-2/3} \Theta'_{n0} e\left(n, -\frac{i\pi}{3} \right)$$

where $n = 1, 2$ and $e(n, x) = \exp\{-(-1)^n x\}$.

ECOULEMENTS GAZ-LIQUIDE A COUCHE LIMITE LAMINAIRE AVEC UN INTERFACE ONDULE DE CHANGEMENT DE PHASE

Résumé—On étudie analytiquement des écoulements de gaz et de liquide à couche limite laminaire avec un interface ondulé de changement de phase, pour déterminer la configuration du champ de perturbation. Le champ de base est approché par des profils linéaires et perturbé par des petites ondes de perturbation. La vitesse de changement de phase à l'interface est perturbé proportionnellement à la puissance 1/3 du nombre d'onde. La relation de phase des contraintes tangentielles et normales à l'interface implique la possibilité d'une instabilité "d'onde liquide" à effet de protection comme dans le cas isotherme, bien que le changement de phase à l'interface agisse pour diminuer cet effet à la fois en terme d'amplitude et de phase.

LAMINARE GAS-FLÜSSIGKEITS-GRENZSCHICHTSTRÖMUNGEN MIT WELLIGER GRENZFLÄCHE

Zusammenfassung—Laminare Grenzschichtströmungen von Gas und Flüssigkeit mit welliger Phasenänderungsgrenzfläche werden analytisch untersucht, um die Eigenschaften ihrer gestörten Schicht zu bestimmen. Die Unterschicht wurde durch lineare Profile mit kleinen, welligen Störungen angenähert. Die Phasenänderungsrate an der Grenzfläche ist gestört proportional zur Potenz 1/3 der Wellenzahl mit der Phasenverschiebung 150° relativ zur Grenzfläche. Die welligen Störungen haben einen um eine Größenordnung kleineren Einfluß auf den Wärmeübergang an der Grenzfläche als auf die Oberflächenreibung. Beide Koeffizienten sind proportional der Potenz 1/3 der Wellenzahl. Der Phasenzusammenhang der Scher- und Normalkräfte enthält dieselbe Möglichkeit der "Wasserwellen"-Instabilität des Schutzeffektes wie für den isothermen Fall, obgleich der Phasenwechsel an der Grenzfläche bezüglich der Amplitude und des Phasenzusammenhangs eine Verminderung eines solchen Effekts bewirkt.

ЛАМИНАРНЫЕ ПОГРАНИЧНЫЕ СЛОИ В ЖИДКОСТИ И ГАЗЕ ПРИ ИЗМЕНЕНИИ ФАЗЫ ВОЛНЫ НА ПОВЕРХНОСТИ РАЗДЕЛА

Аннотация — Для изучения особенностей поля возмущений аналитически исследовались ламинарные пограничные слои в газе и жидкости вблизи поверхности раздела при изменении фазы волны. Основное поле аппроксимировалось линейными профилями и нарушалось небольшими волновыми возмущениями. Скорость изменения фазы волны на поверхности раздела изменяется пропорционально волновому числу в степени 1/3 с отставанием по фазе в 150° относительно поверхности раздела. Волновые возмущения на поверхности раздела сказываются на теплообмене на порядок слабее, чем на поверхностном трении. Оба коэффициента пропорциональны волновому числу в степени 1/3. Соотношение фаз между касательным и нормальным напряжениями на поверхности раздела предполагает, как и в изотермических случаях, нестабильность опрокидывающего эффекта, хотя изменение фаз на поверхности раздела облегчает появление этого эффекта как по амплитуде, так и по фазе.